ASYMMETRY AND PROJECTION CONSTANTS OF BANACH SPACES

BY

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ABSTRACT

We discuss various asymmetry constants of finite-dimensional Banach spaces in a more generalized frame than that of [2], and solve a problem raised in [7] by finding an increasing sequence of Banach spaces whose diagonal asymmetry constants tend to infinity. We investigate the question of whether the projection constant of every n-dimensional Banach space is strictly less than $\sqrt{n}$, and show that this is so when $n = 2$.

1. Introduction

We shall use the concept of normed linear ideals of operators to generalize the results of [2] on asymmetry constants. For a detailed discussion on normed linear ideals of operators we refer the reader to the works of Grothendieck, Pietsch [13] and Schatten.

Let $L(E, F)$ denote the Banach space of linear bounded operators from a Banach space E to a Banach space F. For every pair of Banach spaces E and F, let there be given a norm $\alpha_{E,F}$ defined on a given linear subspace $A(E,F)$ of $L(E,F)$, such that

a) If $u \in A(E,F)$, $v \in L(X,E)$, and $w \in L(F,Y)$, then $wuv \in A(X,Y)$ and $\alpha_{X,Y}(wuv) \leqq \| w \| \, \| v \| \, \alpha_{E,F}(u)$.

b) If $u \in A(E,F)$, then $\alpha(u) \geqq \| u \|$.

c) If $u \in L(E,F)$ is of rank one, then $\alpha_{E,F}(u) = \| u \|$.

The pair $\langle A, \alpha \rangle$ is called a normed linear ideal of operators (N.L.I.O.) c.f. [13]. Given a N.L.I.O. $\langle A, \alpha \rangle$, the conjugate ideal $\langle A^{\Delta}, \alpha^{\Delta} \rangle$ is defined as in [4]: $T \in A^{\Delta}(E,F)$ if and only if there is an $\mathscr{S} > 0$ such that for every finite rank $L \in L(F,E)$ the inequality

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$$|\text{trace }(LT)| \leqq \mathscr{S}\alpha(L),$$

holds, and $\alpha^{\Delta}(T)$ is defined as inf $\mathscr{S}$, taken over all possible $\mathscr{S}$. We shall use the following result:

PROPOSITION 1. *Let $\langle A,\alpha\rangle$ be a N.L.I.O. and E and F be finite-dimensional Banach spaces. Then $(L^{\Delta}(E,F),\alpha^{\Delta})$ is a Banach space which may be identified with the conjugate space $(L(F,E),\alpha)^*$ where the correspondence is given by*

$$\langle S,T\rangle = \text{trace }(ST)$$

for every $S\in(L(F,E),\alpha)$, $T\in(L(E,F),\alpha^{\Delta})$.

Some well-known examples of N.L.I.O.'s which we use here are:

1) (Π_p,π_p) $(1\leqq p\leqq\infty)$ the ideal of p-absolutely summing operators [11, 12, 16].

2) (N_p,ν_p) $(1\leqq p\leqq\infty)$ the ideal of p-nuclear operators [11, 16].

3) (I_p,i_p) $(1\leqq p\leqq\infty)$ the ideal of p-integral operators [11, 16], that is, operators $u\in L(E,F)$ for which there is a probability measure space (Ω,Σ,μ) and operators $v\in L(E,L_\infty(\Omega,\mu))$ and $w\in L(L_p(\Omega,\mu),F'')$ such that $wjv = iu$, where $j\colon L_\infty(\Omega,\mu)\to L_p(\Omega,\mu)$ and $i\colon F\to F''$ are the canonical injections, and where $i_p(u)$ is defined as inf$\|w\|\,\|v\|$ taken over all such factorizations. Observe that if F is reflexive, $i_\infty = c$ which is the extension norm defined for each $u\in L(E,F)$ as the infimum of all λ such that whenever E_0 is a Banach space containing E as a closed linear subspace, u has an extension $\tilde u\in L(E_0,F)$ with norm $\leqq\lambda$.

When either E or F is finite-dimensional, $\Pi_{p'}^{\Delta}(E,F) = N_p(E,F) = I_p(E,F)$ $(1/p+1/p' = 1)$ with equality of norms, in particular $\pi_1^{\Delta} = \nu_\infty = i_\infty = c$ in this case [11, 16].

We recall now some definitions of asymmetry [2]. Let E be a Banach space and G a (multiplicative) group of operators in $L(E,E)$. G' denotes the set of all $t\in L(E,E)$ which commute with every $g\in G$. G_E denotes the set of all isometries of E. We say E has enough symmetries if G_E' consists only of the scalar multiples of the identity operator i_E on E.

The asymmetry constant $s(E)$ is defined as the infimum of all $\lambda>0$ for which there is a group $G\subseteq L(E,E)$ of invertible "onto" operators such that $\sup\{\|g\|;\, g\in G\}\leqq\lambda$, and $G' = \{\alpha i_E\}$.

If E is a Banach space with basis $B = \{e_i\}$, and σ is a finite permutation of the integers, then the operator $g_\sigma\in L(E,E)$ is defined by $g_\sigma(e_i) = e_{\sigma(i)}$, and the diagonal asymmetry constant $\delta(B)$ is defined as $\sup_\sigma\|g_\sigma\|$. If $\varepsilon = (\varepsilon_i)$ is a sequence of ± 1, where $\varepsilon_i = 1$ except for a finite number of i for which $\varepsilon_i = -1$, then $g_\varepsilon \in L(E,E)$ is defined by $g_\varepsilon(e_i) = \varepsilon_i e_i$, and the coordinate asymmetry constant $x(B)$ is defined as $\sup_\varepsilon \| g_\varepsilon \|$. Note that $s(E) \leqq \delta(B)x(B)$ for every basis B [2].

The diagonal asymmetry constant of E, $\delta(E)$, is defined as $\inf\{\delta(B);\ B$ is a basis for $E\}$, and the coordinate (or unconditional) asymmetry constant $x(E)$ as $\inf\{x(B);\ B$ is a basis for $E\}$.

If $\langle A, \alpha \rangle$ is a N.L.I.O. and i_E the identity operator on E, then $\alpha(E)$ will denote $\alpha(i_E)$, $\lambda(E) = c(i_E)$ will denote the projection constant of E. The distance coefficient between isomorphic Banach spaces E and F is defined by $d(E,F) = \inf \| t \| \| t^{-1} \|$, where t is any isomorphism of E onto F.

We consider two problems raised in [7]: Is there a sequence E_n $n = 1, 2, \cdots$ of finite-dimensional real Banach spaces such that

1) $\delta(E_n) \to \infty$?

2) $x(E_n) \to \infty$?

It was shown in [2], that there is a sequence E_n for which $s(E_n) \to \infty$. We prove here, Theorem 4, that for this sequence also $\delta(E_n) \to \infty$. Our results differ from those of [2] mainly because there is no known relationship between the asymmetry constants $\delta(E)$, $x(E)$ and $s(E)$. It is therefore theoretically possible for any one of these constants to be arbitrarily large for suitable E's, while the other two constants remain bounded or small in comparison. However, we do not know of finite-dimensional examples which demonstrate these phenomena.

Theorems 2, 3 and 5 relate $\alpha(E)$, $\alpha^\Delta(E)$ and $\dim(E)$ to each one of the three asymmetry constants. Their applications are useful and we illustrate some of them in Corollaries 3 and 4 and Theorems 4 and 7. In section 3 we discuss the constant $c_n = \max\{\lambda(E);\ \dim(E) = n\}$. It is an open and apparently difficult question whether c_n is strictly less than $\sqrt{n}$ for each $n = 2, 3, \cdots$. We prove in Theorem 8 that $c_2 < \sqrt{2}$; in fact, $c_2 < 1.414211$.

2. Asymmetry constants

THEOREM 1 [2]. *$s(E) = \inf\{d(E,F);\ F$ is a Banach space with enough symmetries$\}$.*

The following is a generalization of [2, Th. 6].

THEOREM 2. *If E is an n-dimensional Banach space, and $\langle L(E,E), \alpha \rangle$ is a N.L.I.O., then*

$$n \leqq \alpha(E)\alpha^\Delta(E) \leqq n(s(E))^2.$$

PROOF. By Proposition 1,

$$\alpha(i_E)\alpha^\Delta(i_E) \geqq \operatorname{trace}(i_E \cdot i_E) = \operatorname{trace}(i_E) = n.$$

Let F be any n-dimensional Banach space with enough symmetries. Since G_F is a compact group, there is a unique normalized positive Haar measure dg on G_F. By Proposition 1, there is an operator $u \in L(F,F)$ with $\alpha^\Delta(u) = 1$ and $\alpha(F) = \operatorname{trace}(u)$.

Let $v = \int_{G_F} g^{-1}ug\,dg$. Since $v \in G'_F$, therefore $v = \lambda i_F$ for some scalar λ, where λ is given by:

$$\alpha(F) = \operatorname{trace}(u) = \operatorname{trace}(v) = \lambda \cdot n.$$

It then follows that

$$n^{-1}\alpha(F)\alpha^\Delta(F) = \alpha^\Delta(v) \leqq \int_{G_F} \alpha^\Delta(g^{-1}ug)dg \leqq \int_{G_F} \| g^{-1} \| a^\Delta(u) \| g \| dg = 1.$$

But $\alpha(E) \leqq d(E,F)\alpha(F)$, and similarly for α^Δ; therefore,

$$\alpha(E)\alpha^\Delta(E) \leqq (d(E,F))^2\alpha(F)\alpha^\Delta(F) = n(d(E,F))^2,$$

and the result follows by Theorem 1. Q.E.D.

COROLLARY 1. *If E has enough symmetries,* $\alpha(E)\alpha^\Delta(E) = n$.

THEOREM 3. *If E is as in Theorem 2, then*

$$n \leqq \alpha(E)\alpha^\Delta(E) \leqq 3n(\delta(E))^3.$$

PROOF. By Proposition 1, there is an operator $u \in L(E,E)$, $\alpha^\Delta(u) = 1$ such that $\alpha(E) = \operatorname{trace}(u)$. Let $B = \{e_i\}_1^n$ be any basis for E, and put $\delta = \delta(B)$. Let

$$v = (n!)^{-1}\Sigma g_\sigma^{-1}ug_\sigma,$$

where σ ranges over all the permutations of $1,2,\cdots,n$. Clearly there exist scalars a, b such that $v = ai_E + bw$, where w is the rank one projection of E onto the space spanned by the vector $e = n^{-1} \Sigma_1^n e_i$, given by $we_i = e$ $(i = 1,2,\cdots,n)$. Now $\alpha(E) = \operatorname{trace}(u) = \operatorname{trace}(v) = an + b$, and therefore $v = ai_E + (\alpha(E) - an)w$. Also

$$\alpha^\Delta(v) \leqq (n!)^{-1} \Sigma \alpha^\Delta(g_\sigma^{-1}ug_\sigma) \leqq (n!)^{-1} \Sigma \| g_\sigma^{-1} \| \| g_\sigma \| \alpha^\Delta(u) \leqq \delta^2.$$

On the other hand,

$$\| w \| \alpha^\Delta(v) \geqq a^\Delta(wv) = |a + \alpha(E) - an| \alpha^\Delta(w) = |a + \alpha(E) - an| \| w \|,$$

so that $\alpha^\Delta(v) \geqq |a + \alpha(E) - an|$. In addition

$$\alpha^\Delta(v) \geqq |a|\alpha^\Delta(i_E) - |\alpha(E) - an|\alpha^\Delta(w) = |a|\alpha^\Delta(E) - |\alpha(E) - an|\,\|w\|,$$

$$\|w(\sum x_i e_i)\| = n^{-1}|\sum x_i|\,\|e_i\| = (n!)^{-1}\ \Big\|\sum_\sigma \sum_i x_i e_{\sigma(i)}\Big\|$$

$$\leqq (n!)^{-1}\sum_\sigma \Big\|\sum_i x_i e_{\sigma(i)}\Big\| \leqq \delta \Big\|\sum x_i e_i\Big\|$$

so that $\|w\| \leqq \delta$, and this implies that $\alpha^\Delta(v) \geqq |a|\alpha^\Delta(E) - |\alpha(E) - an|\delta$.

Combining the inequalities we obtain

$$\delta^2 \geqq \max\{|a(n-1) - \alpha(E)|,\ |a|\alpha^\Delta(E) - |an - \alpha(E)|\delta\}.$$

If we assume $\alpha^\Delta(E) > \delta$, otherwise the proof is complete, then a simple calculation shows that the minimum of the function

$$f(x) = \max\{|x(n-1) - \alpha(E)|,\ |x|\alpha^\Delta(E) - |xn - \alpha(E)|\delta\}$$

in the interval $-\infty < x < \infty$ is the value

$$A = \min_{\varepsilon = \pm 1} \frac{\alpha(E)(\alpha^\Delta(E) + \varepsilon\delta)}{n\delta + n - 1 + \varepsilon\alpha^\Delta(E)},$$

so it follows that $\delta^2 \geqq A$.

If the minimum for A is attained when $\varepsilon = 1$, then

$$(n\delta + n - 1 + \alpha^\Delta(E))\delta^2 \geqq \alpha(E)\alpha^\Delta(E) + \alpha(E)\delta,$$

and since ν_1 is the greatest cross norm [5], therefore $\alpha^\Delta(E) \leqq \nu_1(E) = n$ [2], hence $\alpha^\Delta(E) \leqq n$, from which it follows that

$$3n\delta^3 \geqq (n\delta + n - 1 + \alpha^\Delta(E))\delta^2 \geqq \alpha(E)\alpha^\Delta(E).$$

If the minimum for A is attained when $\varepsilon = -1$, then

$$2n\delta^3 \geqq (n\delta + n - 1 - \alpha^\Delta(E))\delta^2 \geqq \alpha(E)\alpha^\Delta(E) - \alpha(E)\delta \geqq \alpha(E)\alpha^\Delta(E) - n\delta^3,$$

and this concludes the proof. Q.E.D.

COROLLARY 2. *If $\delta(E) = 1$, then $n \leqq \alpha(E)\alpha^\Delta(E) \leqq 3n$.*

Denote by l_p^n $(1 \leqq p \leqq \infty)$ the n-dimensional l_p space and given two Banach spaces E and F, $E \oplus F$ will denote their direct sum normed by $\|(x,y)\| = \max\{\|x\|, \|y\|\}$. Theorem 4 solves problem (1) mentioned earlier.

THEOREM 4. *If $1 \leqq p \neq q \leqq \infty$, then there exists a constant $c_{p,q} > 0$ such that for every n*

$$\delta(l_p^n \oplus l_q^n) \geqq c_{p,q}\begin{cases} n^{|1/3p - 1/3q|}; & \text{if } (p-2)(q-2) \geqq 0 \\ \max\{n^{1/3p-1/6}, n^{1/6-1/3q}\}; & \text{if } q \geqq 2 \geqq p. \end{cases}$$

PROOF. We shall apply Theorem 3 with $\alpha = \pi_1$, the 1-absolutely summing norm, for which we noted above that $\pi_1^\Delta = i_\infty = c$, so that $\pi_1^\Delta(E) = \lambda(E)$.

Let $E_n = l_p^n \oplus l_q^n$. It was shown in [2] that if $(p-2)(q-2) \geqq 0$ then

$$\lambda(E_n)\pi_1(E_n) \sim n^{1+|1/q-1/p|}$$

($\sim$ means that the ratio of both sides is bounded from 0 and ∞ as $n \to \infty$), and that if $q \geqq 2 \geqq p$ then

$$\lambda(E_n)\pi_1(E_n) \sim n^{3/2-1/q}$$

so the result follows by Theorem 3. Q.E.D.

Concerning the asymmetry x we have

THEOREM 5. *Let E be an n-dimensional Banach space with basis $B = \{e_i\}_1^n$ and $\langle L(E,E), \alpha\rangle$ be an N.L.I.O. For any subset $J \subseteq \{1,2,\cdots,n\}$ let $E_J = [e_i;\ i \in J]$ and*

$$a_j = \min\{\max(\alpha^\Delta(E_I);\ I \subseteq J);\ J \text{ contains } j \text{ elements}\}\ j = 1,2,\cdots,n.$$

Then

$$(x(B))^3 \sum_1^n a_j^{-1} \geqq \alpha(E).$$

PROOF. By Proposition 1 there exists $u \in L(E,E)$ with $\alpha^\Delta(u) = 1$ and $\alpha(E) = \text{trace}(u)$. Let $v = 2^{-n} \sum_\varepsilon g_\varepsilon^{-1} u g_\varepsilon$, where ε ranges over all vectors $(\pm 1, \pm 1, \cdots, \pm 1)$, and let $\{e_i'\}$ be the associated sequence of coefficient functionals to $\{e_i\}$. For v thus defined there exist scalars λ_i^0 such that $ve_i = \lambda_i^0 e_i$ for every i. Let $I \subseteq J$ be any subsets of $\{1,2,\cdots,n\}$ and $w_I\colon E \to E_I$ and $v_I\colon E_I \to E$ be the natural projection and embedding operators respectively.

Let $z\colon E_I \to E_I$ be an arbitrary operator. Since $\|w_I\| \leqq x$ (where $x = x(B)$), we get $\alpha(v_I z w_I) \leqq x \|v_I\| \alpha(z) = x\alpha(z)$. In addition, for any $g_{\varepsilon'}$:

$$\alpha^\Delta(vv_J w_J g_{\varepsilon'}) \leqq 2^{-n} \sum_\varepsilon \|g_\varepsilon^{-1}\| \, \|g_\varepsilon v_J w_J g_{\varepsilon'}\| \, a^\Delta(u) \leqq x^2,$$

since $\|g_\varepsilon v_J w_J g_{\varepsilon'}\| \leqq x$. So combining the inequalities

$$\begin{aligned} x^3\alpha(z) &\geqq \sup_{\varepsilon'} \text{trace}(vv_J w_J g_{\varepsilon'} v_I z w_I) \\ &= \sup_{\varepsilon'} \sum_{i \in I} \lambda_i^0 \varepsilon_i' \langle ze_i, e_i'\rangle = \sum_{i \in I} |\lambda_i^0| \, |\langle ze_i, e_i'\rangle| \\ &\geqq (\min_{j \in J} |\lambda_j^0|) \text{ trace}(z), \end{aligned}$$

where $I \subseteq J$ are arbitrary. This implies that $x^3 \geqq (\min_{j\in J} |\lambda_j^0|)\alpha^\Delta(E_I)$, and maximizing over $I \subseteq J$

$$x^3 \geqq (\min_{j \in J} |\lambda_j^0|) \max_{I \subseteq J} \alpha^\Delta(E_I) \geqq (\min_{j \in J} |\lambda_j^0|) a_{|J|}$$

where $|J|$ denotes the number of elements in J, and since also J is arbitrary and

$$\sum_1^n \lambda_i^0 = \operatorname{trace}(v) = \operatorname{trace}(u) = \alpha(E)$$

we finally get

$$x^3 \geqq \max_J \{(\min_{j \in J} |\lambda_j^0|) a_{|J|}\} \geqq \min \left\{ \max_J (a_{|J|} \min_{j \in J} |\lambda_j|);\ \sum_1^n \lambda_i = \alpha(E) \right\}.$$

We may assume without loss of generality that the minimum on the λ_i is attained for $\lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_n$, and then

$$x^3 \geqq \min\{\max_{1 \leqq i \leqq n} a_i \lambda_i;\ \sum_1^n \lambda_i = \alpha(E),\ \lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_n \geqq 0\}$$

and by virtue of $a_{i+1} \geqq a_i$ $(i = 1, \cdots, n-1)$, the expression is minimized when $a_i \lambda_i = a_1 \lambda_1$ for each i, that is

$$a_i \lambda_i = \alpha(E) \left(\sum_1^n a_j^{-1} \right)^{-1},$$

and the theorem is established. Q.E.D.

It was shown in [2] that for any n-dimensional subspace $E \subset l_p$ $(1 \leqq p \leqq 2)$, $\lambda(E) \geqq K_G^{-1} \sqrt{n}$, where K_G is the Grothendieck constant. Thus applying Theorems 2, 3, and 5 we obtain

COROLLARY 3. *Let E be any n-dimensional subspace of l_p $(1 \leqq p \leqq 2)$, and let $\mu(E) = \min\{(s(E))^2, 3(\delta(E))^3, 2(x(E))^3\}$. Then*

$$K_G \mu(E) \sqrt{n} \geqq \pi_1(E) \geqq \sqrt{n}.$$

PROOF. $\pi_1(E) \geqq \sqrt{n}$ by [2], and Theorems 2 and 3 yield that

$$n \min\{(s(E))^2, 3(\delta(E))^3\} \geqq \pi_1(E) \lambda(E) \geqq \pi_1(E) K_G^{-1} \sqrt{n}.$$

Applying Theorem 5 with $\alpha = \pi_1$, we have $\alpha^\Delta = c$, hence $a_j \geqq K_G^{-1} \sqrt{j}$, $j = 1, \cdots, n$, therefore

$$\pi_1(E)(x(B))^{-3} \leqq \sum_1^n a_j^{-1} \leqq K_G \Sigma j^{-1/2} \leqq 2K_G \sqrt{n}$$

from which it follows that $\pi_1(E) \leqq 2K_G(x(E))^3 \sqrt{n}$. Q.E.D.

REMARKS. We do not know whether $\pi_1(E) n^{-1/2}$ is uniformly bounded for every n-dimensional $E \subset l_p$. However, if l_p contains a sequence E_n of n-dimensional subspaces such that $\pi_1(E_n) n^{-1/2} \to \infty$, then $\mu(E_n)$, and in particular, $x(E_n)$, tend to infinity. Corollary 3 tells us also that $\pi_1(E_n) \sim \sqrt{n}$ whenever $\{\mu(E_n)\}$ is bounded. It is in fact easy to construct $E_n \subset l_p$ such that $s(\bar{E_n})$, $\delta(E_n)$ are unknown, or at least difficult to evaluate, but $x(E_n) = 1$ trivially.

It was proved by Kwapien [10], that every map from every $C(S)$ space into L_q is s-absolutely summing up for all $\infty > s > q > 2$. Rosenthal [14] has recently shown that $\pi_s(T) \leqq c_{q\,s} \| T \|$ for all $T \in L(C(S), L_q)$, where

$$c_{q.s} = c_q((q-1)(s-1)/(s-q))^{1-1/s}$$

and c_q depends only on q. Applying this we get

THEOREM 6. *Let E be any n-dimensional subspace of l_q $(2 < q < \infty)$. Then $\lambda(E) \geqq c_{q,s}^{-1} n^{1/s}$ for every s, $q < s < \infty$.*

PROOF. Let $j: E \to C(S)$ be any isometric embedding in a $C(S)$ space, and P be any bounded linear projection of $C(S)$ onto $j(E)$. $j^{-1}P$ maps $C(S)$ into l_q, hence $\pi_s(P) \leqq \pi_s(j^{-1}P) \leqq \| j^{-1}P \| c_{q,s}$. But clearly $\pi_s(P) \geqq \pi_s(E)$, and by [2] $\pi_s(E) \geqq n^{1/s}$, so that $n^{1/s} \leqq \| P \| c_{q,s}$. Q.E.D.

That $\lambda(l_q^n) \sim n^{1/q}$ was proved by Rutovitz [15] (cf. also [2]). Defining $\mu(E)$ as in Corollary 3 we obtain

COROLLARY 4. *Let $2 < q < s < \infty$ and E be any n-dimensional subspace of l_q. Then*

$$\sqrt{n} \leqq \pi_1(E) \leqq c_{q,s}\mu(E)n^{1-1/s}.$$

PROOF. $\pi_1(E) \geqq \sqrt{n}$ by [2], and Theorems 2 and 3 yield that

$$n \min\{(s(E))^2, 3(\delta(E))^3\} \geqq \pi_1(E)\lambda(E) \geqq c_{q,s}^{-1} n^{1/s}\pi_1(E).$$

Again applying Theorem 5 with $\alpha = \pi_1$, we have by Theorem 6 $a_j \geqq c_{q,s}^{-1} j^{1/s}$ $j = 1, 2, \cdots, n$, hence

$$\pi_1(E)(x(B))^{-3} \leqq \sum_1^n a_j^{-1} \leqq c_{q,s}\Sigma j^{-1/s} \leqq 2c_{q,s}n^{1-1/s}$$

which implies $\pi_1(E) \leqq 2c_{q,s}(x(E))^3 n^{1-1/s}$. Q.E.D.

REMARKS. $\pi_1(l_q^n) \sim n^{1-1/q}$ and $\pi_1(l_2^n) \sim n^{1/2}$ [3], and both spaces are isometric to subspaces of $L_q[0,1]$. However, also in this case we do not know whether

$$\sup\{\pi_1(E);\ E \subset l_q,\ \dim(E) = n\} \sim n^{1-1/q}.$$

If l_q contains a sequence E_n of n-dimensional subspaces such that $\pi_1(E_n)n^{1/q-1} \to \infty$ then by Corollary 4, $\mu(E_n)$, and therefore also $x(E_n)$, tend to infinity; for it is clear from the definition of $c_{q,s}$ that there is a sequence $s_n \downarrow q$ such that $\pi_1(E_n)n^{1/s_n-1} c_{q,s}^{-1} \to \infty$.†

† *Added in proof:* It is known that the unconditional basis constant of $L(l_p^n, l_q^n)$ $(1 < p \leqq \infty, 1 \leqq q < \infty)$ tends to ∞ with n [18].

3. Projection constants

Let $c_n = \max\{\lambda(E);\ E$ is a real n-dimensional Banach space$\}$. For $n \geqq 2$, it is known that $c_n \leqq \sqrt{n}$ [2, 9], yet it is unknown whether $c_n = \sqrt{n}$ for some n. We shall show in Theorem 8 that $c_2 < \sqrt{2}$. The proof utilizes John's Theorem [8]. Let us first construct an n-dimensional Banach space which has the largest known projection constant: Let $E(\alpha)$, $1 < \alpha < n$, be the space whose points $x = (x_1, \cdots, x_n)$ are normed by

$$\| x \| = \max \left\{ \max_{1 \leqq i \leqq n} |x_i|,\ \alpha^{-1} \sum_{1}^{n} |x_i| \right\}.$$

THEOREM 7. *If* $1 \leqq p < \infty$, *and* $1/p + 1/p' = 1$, *then*

$$(n^{-1} v_{p'}(E(\alpha)))^p = (\pi_p(E(\alpha)))^{-p}$$

$$= \max_{0 \leqq \mu \leqq 1} \min \{\mu\alpha^{-p} + (1-\mu)n^{-1}, \mu(\pi_p(l_1^n))^{-p} + (1-\mu)\alpha^p n^{-p}\}.$$

PROOF. Let S be the unit ball of $E(\alpha)$. The set of the extremal points of S^*, K^*, consists of all points derived from the two points $R = (1, 0, \cdots, 0)$ and $Q = \alpha^{-1}(1, 1, \cdots, 1)$ by all the possible permutations and changes of signs on their coordinates. We assign to Q, and to each point thus derived from Q, the same positive point mass λ, and we assign to R, and to each point derived from R, the same positive point mass w. The total measure assigned to K^* is then $m(K^*) = 2nw + 2^n\lambda$. By [3]

$$(\pi_p(E(\alpha)))^{-p} = \sup_m \inf_{\|x\|=1} \int_{K^*} |\langle x, a\rangle|^p dm(a)$$

where m ranges on all probability measures on K^* which are invariant to isometries; that is, on all measures m as defined above for Q, R, and their derived points, where $0 \leqq \lambda \leqq 2^{-n}$ and $m(K^*) = 1$. It then follows that

$$(\pi_p(E(\alpha)))^{-p} = \sup_{0 \leqq \lambda \leqq 2^{-n}} \min_{\|x\|=1} \left[\lambda \sum_{|\varepsilon_i|=1} \alpha^{-p} \left| \sum_{i=1}^{n} \varepsilon_i x_i \right|^p + 2w \sum_{i=1}^{n} |x_i|^p \right].$$

Knowing K^*, we have that the equations of the supporting planes to S are $x = \pm 1$, $i = 1, 2, \cdots, n$, and $\sum_{i=1}^{n} \varepsilon_i x_i = \alpha$, $\varepsilon_i = \pm 1$.

Since the function

$$f(x) = \left(\int_{K^*} |\langle x, a\rangle|^p dm(a) \right)^{1/p}$$

is convex, therefore its minimum on $\partial S = \{x;\ \| x \| = 1\}$ is attained in our case at the center of gravity of one of the $(n-1)$-dimensional faces of S which, by applying a suitable isometry, we take to be either on the plane $x_1 = 1$ or on the plane $\sum_1^n x_i = \alpha$.

The center of gravity of the extremal points of S on $x_1 = 1$ is the point $A = (1,0,\cdots,0)$, and denoting $\mu = 2^n\lambda$, we have $f(A) = (\mu\alpha^{-p} + (1-\mu)n^{-1})^{1/p}$.

The point B, the center of gravity of the extremal points of S on $\sum_1^n x_i = \alpha$, is found as follows: If α is an integer, that is $\alpha = [\alpha]$, then the extremal points of S on $\sum_1^n x_i = \alpha$ are all those of the form $(x_1,\cdots,x_n)$ where $x_i \in \{0,1\}$ and $\sum_1^n x_i = \alpha$. There are $\binom{n}{\alpha}$ such points and easy calculation shows that $B = \alpha n^{-1}(1,1,\cdots,1)$.

If $\alpha \neq [\alpha]$, then the extremal points are those derived from

$$(\underbrace{1,1,\cdots,1}_{[\alpha]},\alpha-[\alpha],0,\cdots,0)$$

by all the permutations on the coordinates. There are $n\binom{n-1}{[\alpha]}$ such points, and again one has that $B = \alpha n^{-1}(1,1,\cdots,1)$. We then get

$$\begin{aligned}(f(B))^p &= \lambda \sum_{\varepsilon_i} \alpha^{-p}(\alpha/n)^p \Big| \sum_{i=1}^{n} \varepsilon_i \Big|^p + 2w(\alpha/n)^p\, n \\ &= \mu 2^{-n} n^{-p} \sum_{i=0}^{n} \binom{n}{i} |n-2i|^p + (1-\mu)(\alpha/n)^p \\ &= \mu(\pi_p(l_1^n))^{-p} + (1-\mu)\alpha^p n^{-p}\end{aligned}$$

by virtue of [3]. Using the fact that $\pi_p^\Delta = \nu_{p'}$, and Corollary 1, the assertion of of the theorem follows from

$$\pi_p(E(\alpha)))^{-p} = \max_{0 \leqq \mu \leqq 1} \min\{f(A))^p,\ (f(B))^p\}. \qquad \text{Q.E.D.}$$

Corollary 5. $\lambda(E(\sqrt{n})) = (n - \lambda(l_1^n))/(2\sqrt{n} - \lambda(l_1^n) - 1)$.

Proof. Take $p = 1$ in Theorem 7, and use the facts that $\lambda(E(\alpha)) = \nu_\infty(E(\alpha))$, and $\pi_1(l_1^n)\lambda(l_1^n) = n$. Q.E.D.

Corollary 6. $\lim_{n\to\infty} \lambda(E(\sqrt{n}))/\sqrt{n} = (2 - \sqrt{2/\pi})^{-1} \approx 0.832$.

Proof. Use Corollary 5 and the result $\lambda(l_1)n^{-1/2} \to_{n\to\infty} \sqrt{2/\pi}$ [6]. Q.E.D.

Note that $(2 - \sqrt{2/\pi})^{-1}$ is larger than $\sqrt{2/\pi}$ which is attained for the space l_1^n (and l_2^n). Our main result in this section is

Theorem 8. $4/3 \leqq c_2 < \sqrt{2}$.

That $c_2 \geqq 4/3$ is due to the fact that the projection constant of the space whose unit ball is the regular hexagon in the plane is 4/3 [2, 3]. To prove the other inequality we need two lemmas. Lemma 1 is due to John [8], and was written in this form in [2].

LEMMA 1. *Let F be a real n-dimensional Banach space with unit ball S. Let $\|\cdot\|_2$ be the Hilbert norm on F with the property that the unit ball B_2 in $(F, \|\cdot\|_2)$ is the ellipsoid of least volume containing S. Then there exist $s \leqq n(n+1)/2$ distinct points $x^1, x^2 \cdots, x^s$ in F, and positive scalars $\lambda_1, \lambda_2, \cdots, \lambda_s$ such that*

1) $\|x^r\|_F = \|x^r\|_{F'} = \|x^r\|_2 = 1$ *for each* $r = 1, \cdots, s$.

2) $x = \sum_1^s \lambda_r \langle x, x^r \rangle x^r$ *for each* $x \in F$ ($\langle \, , \rangle$ *is the inner product defined by* $\|\cdot\|_2$).

3) $\sum_1^s \lambda_r = n$.

4) $x^i \neq -x^j$ *if* $i \neq j$.

LEMMA 2. *Let E be a real 2-dimensional Banach space with $\lambda(E) = \sqrt{2}$ and let $F = E'$. Then under the conditions of Lemma 1, $s = 2$ and there exists $y_0 \in \partial B^* \cap \partial B_2$ (∂B^* is the boundary of the unit ball of $E' = F$) such that $|\langle y_0, x^1 \rangle| = |\langle y_0, x^2 \rangle| = 1/\sqrt{2}$.*

PROOF. Embed E isometrically in l_∞. Then, for every projection $P: l_\infty \to E$, $\|P\| \geqq \lambda(E) = \sqrt{2}$. Since $\|x^i\|_{E'} = 1$ there exist Hahn-Banach extensions $\tilde{x}^i \in (l^\infty)'$ of x^i having norms $= 1$. Define the projection $P: l_\infty \to E$ by

$$Px = \sum_{r=1}^{s} \lambda_r \langle x, \tilde{x}^r \rangle x^r, \quad x \in l_\infty$$

where x^r are viewed as points in E. Let $y_0 \in \partial B^*$ be such that

$$\sum \lambda_r |\langle x^r, y_0 \rangle| = \max\{\sum \lambda_r |\langle x^r, y \rangle|;\ y \in \partial B^*\}.$$

By Lemma 1, Hölder's inequality and the fact that $B_2 \supset B^*$, it follows that

$$\begin{aligned}\sqrt{2} &= \lambda(E) \leqq \sup\{\langle Px, y \rangle;\ \|x\|_\infty = 1,\ y \in \partial B^*\} \\ &= \sup\{\sum \lambda_i \langle x, \tilde{x}^i \rangle \langle x^i, y \rangle;\ \|x\|_\infty = 1,\ y \in \partial B^*\} \\ &\leqq \sup\{\sum \lambda_i |\langle x^i, y \rangle|;\ y \in \partial B^*\} = \sum \lambda\, |\langle x^i, y_0 \rangle| \\ &\leqq (\sum \lambda_i)^{1/2} (\sum \lambda_i \langle x^i, y_0 \rangle^2)^{1/2} = \sqrt{2}\|y_0\|_2\end{aligned}$$

$$\leqq \sqrt{2}\| y_0 \|_{E'} = \sqrt{2}.$$

The equality in Hölder's inequality implies that $|\langle x^i, y_0\rangle| = c$ for each $i = 1, \cdots s$, and that $\| y_0 \|_2 = 1$. Hence

$$1 = \langle y_0, y_0\rangle = \Sigma \lambda_i \langle x^i, y_0\rangle^2 = (\Sigma \lambda_i)c^2 = 2c^2;$$

therefore, $c = 2^{-1/2}$. Since $|\langle x^i, y_0\rangle| = 1/\sqrt{2}$ and the x^i do not all lie on a straight line through the origin, we obtain from (4) that $s = 2$. Q.E.D.

By a suitable rotation of the x, y axes and replacing x^i by $-x^i$ if necessary, we may and shall assume henceforth that $y_0 = (1,0)$ and $x^1 = (1/\sqrt{2}, -1/\sqrt{2})$ and $x^2 = (1/\sqrt{2}, 1/\sqrt{2})$.

Proof of Theorem 8. Assume to the contrary that $\lambda(E) = \sqrt{2}$ for some space E. Let y_0, x^i $(i = 1,2)$ be as in Lemma 2, and let the boundary ∂B of the unit ball of E intersect the positive y-axis at $T = (0,t)$, where $t \geqq 1$ (since $B \supseteq B_2$). Since $\| x^i \|_2 = \| x^i \|_E = 1$ and $B \supseteq B_2$, therefore the tangent line l_i to the circle B_2 drawn through the point x^i $(i = 1,2)$ supports ∂B. Since $y_0 \in \partial B^* \cap \partial B_2$, therefore $y_0 \in \partial B$. Let h be the tangent line to B_2 through y_0. Let $P_i = (1, (-1)^i \operatorname{tg}(\pi/8))$ $(i = 1,2)$ be the point of intersection of l_i and h, and g_1 (resp., g_2) be the tangent line to B_2 drawn through $-T = (0,-t)$ (resp., T) which meets B_2 on the left side of the y-axis, and finally, let $Q_i = l_i \cap g_i$ $(i = 1,2)$.

Since $B \supseteq B_2$ it follows that l_1, l_2, and h all support B, and therefore B is contained in the convex hull of the set $A = \{\pm T, \pm P_1, \pm P_2, \pm Q_1, \pm Q_2\}$. Let Γ be the parallelogram whose vertices are $\pm y_0$ and $\pm T$, and α be the angle $\measuredangle O y_0 T$. Clearly $\Gamma \subseteq B$. We intend to find a number $\beta > 0$ such that $\beta\Gamma \supseteq B$. Convex $(A) \supseteq B$; therefore, if $\beta > 0$ is the least positive number such that $\beta\Gamma \supseteq$ convex (A), then at least one point of the set $\{\pm Q_1, \pm Q_2, \pm P_1, \pm P_2\}$ belongs to the boundary $\partial(\beta\Gamma)$ of $\beta\Gamma$. Since $\alpha \geqq \pi/4$, therefore $P_2 \in \partial(\beta\Gamma)$ from which it follows immediately that $\partial(\beta\Gamma)$ intersects the positive x-axis at the point $\beta = 1 + \operatorname{tg}(\pi/8)\operatorname{ctg}\alpha$.

$\beta\Gamma \supseteq B \supseteq \Gamma$ implies that $\beta \geqq d(E, l^2_\infty) \geqq \lambda(E) = \sqrt{2}$, that is $\operatorname{tg}(\pi/8)\operatorname{ctg}\alpha \geqq \sqrt{2} - 1 = \operatorname{tg}(\pi/8)$; therefore, $\alpha \leqq \pi/4$, which implies that $\alpha = \pi/4$ and $T = (0,1)$, and so $Q_i = (\operatorname{tg}(\pi/8), (-1)^i)$ $(i = 1,2)$. Therefore $A \subseteq (\sec(\pi/8))B_2$, and so $(\sec(\pi/8))B_2 \supseteq B \supseteq B_2$, hence $\sec(\pi/8) \geqq d(E, l^2_2) \geqq \sqrt{2}/\lambda(l^2_2)$, however $\lambda(l^2_2) = 4/\pi$ [6], and this results in the contradiction $\sec(\pi/8) \geqq (\pi\sqrt{2})/4$. Q.E.D.

Remark. A finer argument also based on Lemma 1 shows that $c_2 < 1.414211$, however, the proof is much more complicated and we omit it. As in [1] or [2], it follows that if $E \supset F$ are Banach spaces and $\dim(E/F) = 2$, then there is a projection P of E onto F with norm $< 2.414211 < (1 + \sqrt{2})$.

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